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From the Becker–Döring to the Lifshitz–Slyozov–Wagner Equations

Philippe Laurençot[1] and Stéphane Mischler[2]

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Connections between two classical models of phase transitions, the Becker–Döring (BD) equations and the Lifshitz–Slyozov–Wagner (LSW) equations, are investigated. Homogeneous coefficients are considered and a scaling of the BD equations is introduced in the spirit of the previous works by Penrose and Collet, Goudon, Poupaud and Vasseur. Convergence of the solutions to these rescaled BD equations towards a solution to the LSW equations is shown. For general coefficients an approach in the spirit of numerical analysis allows to approximate the LSW equations by a sequence of BD equations. A new uniqueness result for the BD equations is also provided.

KEY WORDS: Asymptotics; Becker–Döring equations; kinetics of phase transitions; Lifshitz–Slyozov–Wagner equation; uniqueness

1. INTRODUCTION

The Becker–Döring (BD) equations(1) and the Lifshitz–Slyozov–Wagner (LSW) equations(2, 3) are two classical models of phase transitions describing different stages of the growth of grains of a new phase from a supersaturated solution. The LSW equations actually describe the late stages of the growth process in which no new grain can form. The determining process is then the growth of the grains by, e.g., diffusional mass exchange:(2) the grains of the new phase larger than some critical size grow at the expense

[1] Mathématiques pour l'Industrie et la Physique, CNRS UMR 5640, Université Paul Sabatier, Toulouse 3, 118 route de Narbonne, F-31062 Toulouse cedex 4, France; e-mail: laurencot@mip.ups-tlse.fr

[2] Laboratoire de Mathématiques Appliquées, Université de Versailles, Saint Quentin, 45 avenue des Etats-Unis, F-78035 Versailles, France and DMA, CNRS UMR 8553, Ecole Normale Supérieure, 45 rue d'Ulm, F-75230 Paris cedex 05, France; e-mail: mischler@dma.ens.fr

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of smaller ones, the critical size varying in time as a function of the degree of supersaturation. A mean-field model has been formulated by Lifshitz and Slyozov[2] and Wagner[3] which consists of a continuity equation for the volume distribution function f of the grains coupled with the equation of the conservation of matter. For spherical grains the continuity equation reads

$$\partial_t f + \partial_x(\mathscr{V} f) = 0, \qquad (t, x) \in \mathbb{R}_+^2 \tag{1.1}$$

where $x \in \mathbb{R}_+ := (0, +\infty)$ is the volume of the grains, $t \in \mathbb{R}_+$ is the time variable and $\mathscr{V} = \mathscr{V}(t, x)$ denotes the rate of growth of the grains which is determined by the mechanism of mass transfer between the grains, e.g., volume diffusion[2] or grain-boundary diffusion.[4] In general one has

$$\mathscr{V}(t, x) = k(x)\, u(t) - q(x) \tag{1.2}$$

where the functions k and q are computed from the modeling of the mechanism of mass transfer between the grains.[2–4] Finally, the evolution of the degree of supersaturation $u = u(t)$ is prescribed by the equation of conservation of matter: u is given either by[2]

$$u(t) + A \int_0^\infty x f(t, x)\, dx = Q, \qquad t \geqslant 0 \tag{1.3}$$

where Q is the total initial supersaturation and A is a positive geometric factor, or by[3]

$$u(t) \int_0^\infty k(x)\, f(t, x)\, dx = \int_0^\infty q(x)\, f(t, x)\, dx, \qquad t \geqslant 0 \tag{1.4}$$

Observe that the condition (1.4) is equivalent to the requirement that the solution f to the continuity equation (1.1) satisfies

$$\int_0^\infty x f(t, x)\, dx = \text{const.}, \qquad t \geqslant 0 \tag{1.5}$$

On the other hand the BD equations describe earlier stages of the growth of the grains at a smaller scale (before the grains reach a "macroscopic" size) and have been proposed as a model for the dynamics of a system of clusters of particles which may either gain (coagulation) or shed one particle (fragmentation).[1, 5] If $c_i(t)$ denotes the number of clusters made of i particles (or i-clusters), $i \in \mathbb{N} \setminus \{0\}$, per unit volume at time t, the BD equations read

$$\frac{dc_1}{dt} = -J_1(c) - \sum_{i=1}^{\infty} J_i(c), \qquad t \in \mathbb{R}_+ \tag{1.6}$$

$$\frac{dc_i}{dt} = J_{i-1}(c) - J_i(c), \qquad t \in \mathbb{R}_+, \qquad i \geqslant 2 \tag{1.7}$$

where $c = (c_i)_{i \geqslant 1}$ and

$$J_i(c) = a_i c_1 c_i - b_{i+1} c_{i+1}, \qquad i \geqslant 1 \tag{1.8}$$

Observe that there is no source nor sink of particles in the above model so that the total number of particles is expected to be conserved through time evolution, that is,

$$\sum_{i=1}^{\infty} i c_i(t) = \text{const.}, \qquad t \in [0, +\infty) \tag{1.9}$$

Solutions to (1.6), (1.7) actually enjoy this property.[6]

Since the above mentioned models describe similar phenomena but at different scales it is natural to look for connections between them. Preliminary investigations were performed in ref. 7 and indicate that the behaviour of the distribution of large clusters at large times resulting from the BD equations is approximatively given by the LSW equations. A more precise relationship between the BD and LSW equations has been recently provided by Penrose:[8] introducing a small parameter an asymptotic expansion of suitably rescaled solutions to the BD equations is performed in ref. 8, Section 6 (see also the survey paper of ref. 9) with

$$a_i = a_1 i^{1/3} \qquad \text{and} \qquad b_i = a_1 i^{1/3} (z_s + q i^{-1/3}), \qquad i \geqslant 2$$

a_1, z_s and q being positive real numbers. The lower order term of the expansion is shown to obey the LSW equations (1.1), (1.4) with $k(x) = a_1 x^{1/3}$ and $q(x) = a_1 q$. The arguments of Penrose[8] are however formal and a different scaling of the BD equations leading to the LSW equations (1.1), (1.3) has been proposed more recently in ref. 10 together with a convergence proof. More precisely, given bounded coefficients (a_i) and (b_i) satisfying

$$|a_{i+1} - a_i| + |b_{i+1} - b_i| \leqslant K/i, \qquad i \geqslant 1$$

a suitable rescaling of the BD equations (1.6), (1.7) is introduced in ref. 10 and a subsequence of the solutions to these rescaled BD equations is shown to converge towards a measure-valued solution to the LSW equations (1.1), (1.3). Unfortunately the analysis of ref. 10 does not allow in general to identify the functions k and q defining the growth rate $\mathscr{V}$ in (1.2) except when a_i and b_i both have a limit as $i \to +\infty$.

The aim of this paper is twofold: on the one hand, we return to the approach of refs. 8 and 10 and consider the BD equations with homogeneous coefficients $a_i = i^\lambda$ and $b_i = i^\mu$ for $i \geqslant 1$ and $0 \leqslant \mu < \lambda \leqslant 1$. Rescaling appropriately the BD equations allows us to show the convergence of the corresponding rescaled solutions towards a solution of the LSW equations (1.1), (1.4) with $k(x) = x^\lambda$ and $q(x) = x^\mu$, $x \in \mathbb{R}_+$. More precisely, denoting by $c = (c_i)$ the solution to the BD equations (1.6), (1.7) and introducing a small parameter $\varepsilon \in (0, 1)$ we follow the approach of refs. 8 and 10 and look for the large time behaviour of c by introducing $c^\varepsilon(t) = c(t\varepsilon^{-\nu})$ for $t \in [0, +\infty)$ where ν is to be determined later (the choice $\nu = 1$ is made in refs. 8 and 10). By (1.6), (1.7) and (1.9), $c^\varepsilon = (c_i^\varepsilon)$ is a solution to

$$\frac{dc_i^\varepsilon}{dt} = \varepsilon^{1-\mu-\nu} \left(\varepsilon^{\mu-\lambda} c_1^\varepsilon \, \frac{((i-1)\,\varepsilon)^\lambda \, c_{i-1}^\varepsilon - (i\varepsilon)^\lambda \, c_i^\varepsilon}{\varepsilon} \right) + \varepsilon^{1-\mu-\nu} \left(\frac{((i+1)\,\varepsilon)^\mu \, c_{i+1}^\varepsilon - (i\varepsilon)^\mu \, c_i^\varepsilon}{\varepsilon} \right) \tag{1.10}$$

for $i \geqslant 2$ and

$$c_1^\varepsilon(t) + \sum_{i=2}^{\infty} i\varepsilon^2 \, \frac{c_i^\varepsilon(t)}{\varepsilon^2} = \text{const.} \tag{1.11}$$

First, in order to interpret the series in (1.11) as the first moment of a function f^ε which is piecewise constant on the intervals $[(i-1/2)\,\varepsilon, (i+1/2)\,\varepsilon)$ we are led to consider

$$f^\varepsilon(t, x) = \begin{cases} \dfrac{c_i^\varepsilon(t)}{\varepsilon^2} & \text{if} \quad x \in [(i-1/2)\,\varepsilon, (i+1/2)\,\varepsilon) \quad \text{and} \quad i \geqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

Next, for (1.10) to be close to (1.1) as $\varepsilon \to 0$ the choices $\nu = 1 - \mu$ and $u^\varepsilon = \varepsilon^{\mu-\lambda} c_1^\varepsilon$ yield

$$\partial_t f^\varepsilon \sim -\partial_x((x^\lambda u^\varepsilon - x^\mu)\, f^\varepsilon)$$

while (1.11) reads

$$\varepsilon^{\lambda-\mu} u^{\varepsilon}(t) + \int_0^{\infty} x f^{\varepsilon}(t, x)\, dx = \text{const.} \tag{1.12}$$

Since $\lambda > \mu$ we see at least formally that we may expect $(f^{\varepsilon}, u^{\varepsilon})$ to converge towards a solution to the LSW equations (1.1), (1.4). This expectation turns out to be true and is stated precisely in the next section and proved in Section 5.

On the other hand, we present a different approach to study the relationship between the BD equations (1.6), (1.7) and the LSW equations (1.1), (1.3) which is more in the spirit of numerical analysis and is inspired by refs. 11 and 12. More precisely, in contrast with the previous approach, we start from the LSW equations (1.1), (1.3) and, given functions k and q and a mesh size $\Delta \in (0, 1)$, we construct sequences (a_i^{Δ}) and (b_i^{Δ}) of coefficients for the BD equations. We then use the corresponding solutions to the BD equations to construct a sequence of piecewise constant functions which converges towards a solution to the LSW equations (1.1), (1.3). To give a rough idea of our construction, we consider $\Delta \in (0, 1)$ and denote by $c^{\Delta} = (c_i^{\Delta})$ the solution to the BD equations (1.6), (1.7) with coefficients (a_i^{Δ}) and (b_i^{Δ}) to be determined in terms of k and q. Then

$$\frac{dc_i^{\Delta}}{dt} = c_1^{\Delta} \frac{\Delta a_{i-1}^{\Delta} c_{i-1}^{\Delta} - \Delta a_i^{\Delta} c_i^{\Delta}}{\Delta} + \frac{\Delta b_{i+1}^{\Delta} c_{i+1}^{\Delta} - \Delta b_i^{\Delta} c_i^{\Delta}}{\Delta} \tag{1.13}$$

for $i \geqslant 2$ and

$$c_1^{\Delta}(t) + \sum_{i=2}^{\infty} i\Delta^2 \frac{c_i^{\Delta}(t)}{\Delta^2} = \text{const.} \tag{1.14}$$

Arguing as above we are led to define

$$f^{\Delta}(t, x) = \begin{cases} \dfrac{c_i^{\Delta}(t)}{A\Delta^2} & \text{if} \quad x \in [(i-1/2)\,\Delta, (i+1/2)\,\Delta) \quad \text{and} \quad i \geqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

Putting $u^{\Delta} = c_1^{\Delta}$ we formally recover (1.3) from (1.14) while it is easily seen that (1.13) gives (1.1) in the limit $\Delta \to 0$ if

$$\Delta a_i^{\Delta} \sim k \qquad \text{and} \qquad \Delta b_i^{\Delta} \sim q \qquad \text{on} \quad [(i-1/2)\,\Delta, (i+1/2)\,\Delta)$$

A possible choice for (a_i^{Δ}) and (b_i^{Δ}) is then

$$a_i^{\Delta} = \frac{1}{\Delta^2} \int_{(i-1/2)\,\Delta}^{(i+1/2)\,\Delta} k(x)\, dx \qquad \text{and} \qquad b_i^{\Delta} = \frac{1}{\Delta^2} \int_{(i-1/2)\,\Delta}^{(i+1/2)\,\Delta} q(x)\, dx$$

for $i \geqslant 2$ and a rigorous justification of the above formal computations is performed in Section 4.

We now describe the contents of the paper: the approximations outlined above are stated precisely in the next section, and our main results as well. Besides the convergence of the BD equations towards the LSW equations (1.1), (1.3) and (1.1), (1.4) which are stated in Theorems 2.2 and 2.5, respectively, we also obtain a new uniqueness result for the BD equations (Theorem 2.1) which generalizes a previous result by Ball *et al.*[6] Section 3 is devoted to the proof of the uniqueness of solutions to the BD equations stated in Theorem 2.1. The proofs of the convergence results are performed in Sections 4 and 5, respectively. It is worth mentioning at this point that, though the approximating BD equations are built in a different way, the convergence proofs have some common features and proceed in two steps. We first prove the boundedness of (u^ε) and (u^Δ) which is straightforward for the latter but more difficult to obtain for the former and requires to improve a device from ref. 13. Once the boundedness of (u^ε) and (u^Δ) is shown, the weak compactness in L^1 of (f^ε) and (f^Δ) follows by a suitable adaptation of arguments developed in ref. 14 for the analysis of the LSW equations (1.1), (1.3). Loosely speaking, the main idea behind this part of the proof is the stability with respect to the weak topology of $L^1(\mathbb{R}_+; x\,dx)$ of perturbations of the LSW equations. One then realizes that, once written in terms of (f^ε) or (f^Δ), the approximating BD equations are indeed suitable perturbations of the LSW equations from which the weak compactness of (f^ε) and (f^Δ) follows.

2. MAIN RESULTS

Since the seminal paper,[6] several features of the initial value problem for the Becker–Döring equations (1.6), (1.7) with initial data

$$c_i(0) = c_i^{in}, \qquad i \geqslant 1 \tag{2.1}$$

have been studied (well-posedness[6], convergence to equilibrium[15, 6, 16], metastable states[17]) and we recall now the existence and uniqueness results from ref. 6 we need in the sequel. We are actually able to improve the uniqueness result of ref. 6 so that the uniqueness statement of Theorem 2.1 below is new. We first introduce some notations: we define the space X by

$$X = \left\{ c = (c_i)_{i \geqslant 1} \in \mathbb{R}^{\mathbb{N}\setminus\{0\}}, \sum_{i=1}^{\infty} i\,|c_i| < \infty \right\}$$

which is a Banach space with the norm

$$\|c\|_X = \sum_{i=1}^{\infty} i\,|c_i|, \qquad c \in X$$

and we denote by X^+ the positive cone of X, that is,

$$X^+ = \{c = (c_i)_{i \geqslant 1} \in X,\, c_i \geqslant 0 \text{ for } i \geqslant 1\}$$

Theorem 2.1. Consider $c^{in} = (c_i^{in})_{i \geqslant 1} \in X^+$ and assume that the kinetic coefficients (a_i) and (b_i) enjoy the following properties :

$$0 \leqslant a_i \qquad \text{and} \qquad a_{i+1} - a_i \leqslant K, \qquad i \geqslant 1 \tag{2.2}$$

$$0 \leqslant b_i \qquad \text{and} \qquad b_i - b_{i+1} \leqslant K, \qquad i \geqslant 2 \tag{2.3}$$

for some positive constant K. Then there is a unique function c: $[0, +\infty) \to X^+$ such that

$$c_i \in \mathscr{C}([0, +\infty)) \qquad \text{for} \quad i \geqslant 1 \tag{2.4}$$

$$\sum_{i=1}^{\infty} a_i c_i \in L^1(0, t), \qquad \sum_{i=2}^{\infty} b_i c_i \in L^1(0, t) \tag{2.5}$$

$$c_i(t) = c_i^{in} + \int_0^t (J_{i-1}(c(s)) - J_i(c(s)))\, ds, \qquad i \geqslant 2 \tag{2.6}$$

$$c_1(t) = c_1^{in} - \int_0^t \left(J_1(c(s)) + \sum_{i=1}^{\infty} J_i(c(s)) \right) ds \tag{2.7}$$

and

$$\|c(t)\|_X = \|c^{in}\|_X \tag{2.8}$$

for each $t \in [0, +\infty)$.

Observe first that, owing to (2.8), the integral Eq. (2.7) may be replaced by the algebraic equation

$$c_1(t) + \sum_{i=2}^{\infty} i c_i(t) = \|c^{in}\|_X, \qquad t \in [0, +\infty) \tag{2.9}$$

Notice also that (2.2) implies that

$$0 \leqslant a_i \leqslant i \max\{K, a_1\}, \qquad i \geqslant 1 \tag{2.10}$$

while (2.3) ensures that

$$0 \leqslant B_i \leqslant B_{i+1} \qquad \text{where} \quad B_i := b_i + Ki, \qquad i \geqslant 2 \tag{2.11}$$

On the one hand, owing to (2.10) the existence of a solution c to (1.6), (1.7), (2.1) satisfying the properties stated in Theorem 2.1 follows from ref. 6, Corollaries 2.3 and 2.6. On the other hand the uniqueness results of ref. 6 require either a stronger assumption on the initial datum or that the kinetic coefficients satisfy

$$0 \leqslant a_i + b_i \leqslant C i^{2/3} \qquad \text{with} \quad |a_i - a_{i-1}| + |b_{i+1} - b_i| \leqslant C i^{-1/3}, \qquad i \geqslant 2$$

for some positive constant C, which is clearly stronger than (2.2) and (2.3). Theorem 2.1 thus provides the uniqueness of solutions to (1.6), (1.7), (2.1) for a larger class of kinetic coefficients.

We now turn to the Lifshitz–Slyozov–Wagner equations

$$\partial_t f + \partial_x (\mathscr{V} f) = 0, \qquad (t, x) \in \mathbb{R}_+^2 \tag{2.12}$$

$$f(0, x) = f^{in}(x), \qquad x \in \mathbb{R}_+ \tag{2.13}$$

where the growth rate of the grains is defined by

$$\mathscr{V}(t, x) = k(x)\, u(t) - q(x), \qquad (t, x) \in \mathbb{R}_+^2 \tag{2.14}$$

and u is given either by ref. 2

$$u(t) + A \int_0^\infty x f(t, x)\, dx = Q, \qquad t \in \mathbb{R}_+ \tag{2.15}$$

with $A > 0$ and $Q > 0$, or ref. 3

$$u(t) \int_0^\infty k(x)\, f(t, x)\, dx = \int_0^\infty q(x)\, f(t, x)\, dx, \qquad t \in \mathbb{R}_+ \tag{2.16}$$

As already mentioned the functions k and q involved in the definition of $\mathscr{V}$ are determined by the mechanism of mass transfer between the grains and we provide now a couple of examples which have been derived in refs. 2 and 3, respectively:

— the Lifshitz–Slyozov case:[2] the functions k and q are given by

$$k(x) = 3x^{1/3} \qquad \text{and} \qquad q(x) = 3, \qquad x \in \mathbb{R}_+ \tag{2.17}$$

— the Wagner case:[3] the functions k and q are given by

$$k(x)=\frac{ax^{2/3}}{cx^{1/3}+d} \qquad \text{and} \qquad q(x)=\frac{bx^{1/3}}{cx^{1/3}+d}, \qquad x\in\mathbb{R}_+ \tag{2.18}$$

where a, b, c and d are positive real numbers.

A more thorough description of the computation of the growth rate $\mathscr{V}$ together with the physical assumptions leading to various formulae for k and q may be found in ref. 4.

Our first result deals with the convergence of the BD equations (1.6), (1.7), (2.1) towards the LSW equations (2.12), (2.13), (2.15). It is actually valid for a large class of functions k and q which includes the examples (2.17) and (2.18) and the model case $k(x)=x^\lambda$, $q(x)=x^\mu$ with $0\leqslant\mu<\lambda\leqslant 1$. More precisely we assume that k and q enjoy the following properties.

$$\begin{cases}\text{The function } k \text{ is a non-negative function in } \mathscr{C}([0,+\infty))\cap\mathscr{C}^1(\mathbb{R}_+)\\ \text{satisfying } k'\in L^\infty(1,+\infty) \text{ and } k'\geqslant 0.\end{cases} \tag{2.19}$$

$$\begin{cases}\text{The function } q \text{ is a non-negative and concave function in}\\ \mathscr{C}([0,+\infty))\cap\mathscr{C}^1(\mathbb{R}_+) \text{ satisfying } q'\in L^\infty(1,+\infty) \text{ and } q'\geqslant 0\end{cases} \tag{2.20}$$

In other words the functions k and q are Lipschitz continuous functions for large values of x and might be less regular near $x=0$, but are non-decreasing. We also assume that, for every $U\geqslant 0$, there exists $x_U\in(0,1]$ such that

$$Uk(x)-q(x)\leqslant -xq'(x), \qquad x\in(0,x_U] \tag{2.21}$$

In particular, q' being non-negative by (2.20), we infer from (2.19)–(2.21) that $Uk(0)-q(0)\leqslant 0$ for $U\geqslant 0$ which ensures that no boundary condition is needed at $x=0$ to solve (2.12), (2.13). In addition it follows from (2.20) and (2.21) that

$$\lim_{x\to 0}\frac{q(x)}{k(x)}=+\infty \tag{2.22}$$

We next consider two positive real numbers Q and A and assume that the initial datum f^{in} satisfies

$$\begin{cases} f^{in}\in L^1(\mathbb{R}_+;(1+x)\,dx), \qquad f^{in}\geqslant 0 \quad \text{a.e. in } \mathbb{R}_+ \text{ and}\\ A\int_0^\infty xf^{in}(x)\,dx\leqslant Q\end{cases} \tag{2.23}$$

For $\Delta \in (0, 1)$ and $i \geqslant 2$ we put $\Lambda_i = [(i-1/2)\,\Delta, (i+1/2)\,\Delta)$, $\chi_i^\Delta = \mathbf{1}_{\Lambda_i}$ and

$$a_i^\Delta = \frac{1}{\Delta^2} \int_{\Lambda_i} k(x)\,dx \qquad \text{and} \qquad b_i^\Delta = \frac{1}{\Delta^2} \int_{\Lambda_i} q(x)\,dx \tag{2.24}$$

$$c^{in,\Delta} = (c_i^{in,\Delta}) \qquad \text{with} \quad c_i^{in,\Delta} = A\Delta \int_{\Lambda_i} f^{in}(x)\,dx \tag{2.25}$$

We also put

$$a_1^\Delta = \Delta \int_{\Delta/2}^{3\Delta/2} k(x)\,dx \qquad \text{and} \qquad c_1^{in,\Delta} = Q - A \int_0^\infty x f^{in}(x)\,dx \tag{2.26}$$

Clearly, since

$$\sum_{i=2}^\infty i c_i^{in,\Delta} \leqslant 2 \int_0^\infty x f^{in}(x)\,dx$$

it follows from (2.23) that $c^{in,\Delta}$ belongs to X^+ while (2.19) and (2.20) entail that (a_i^Δ) and (b_i^Δ) satisfy (2.2) and (2.3) (with a constant K possibly depending on Δ). Consequently, by Theorem 2.1 there is a solution $c^\Delta = (c_i^\Delta)$ to the BD equations (1.6), (1.7), (2.1) with kinetic coefficients (a_i^Δ), (b_i^Δ) and initial data $c^{in,\Delta}$. With these notations our first convergence result reads as follows.

Theorem 2.2. For $\Delta \in (0, 1)$ we denote by (f^Δ, u^Δ) the functions defined by

$$f^\Delta(t, x) = \sum_{i=2}^\infty \frac{c_i^\Delta(t)}{A\Delta^2}\, \chi_i^\Delta(x), \qquad (t, x) \in [0, +\infty) \times [0, +\infty) \tag{2.27}$$

$$u^\Delta(t) = c_1^\Delta(t), \qquad t \in [0, +\infty) \tag{2.28}$$

There are a sequence (Δ_n) of real numbers in $(0, 1)$, $\Delta_n \to 0$, and a couple of non-negative functions (f, u) such that

$$\begin{cases} f^{\Delta_n} \longrightarrow f & \text{in} \quad \mathscr{C}([0, t]; w-L^1(\mathbb{R}_+; x\,dx)) \\ u^{\Delta_n} \longrightarrow u & \text{in} \quad \mathscr{C}([0, t]) \end{cases} \tag{2.29}$$

where

$$\begin{cases} f \in \mathscr{C}([0, t]; L^1(\mathbb{R}_+; x\,dx)) \cap L^\infty(0, t; L^1(\mathbb{R}_+)) \\ 0 \leqslant Q - A \displaystyle\int_0^\infty x f(t, x)\,dx = u(t) \end{cases} \tag{2.30}$$

and

$$\int_0^\infty f(t,x)\,g(x)\,dx = \int_0^\infty f^{in}(x)\,g(x)\,dx + \int_0^t \int_0^\infty g_x(x)\,\mathscr{V}(s,x)\,f(s,x)\,dx\,ds \tag{2.31}$$

for each $t \in \mathbb{R}_+$ and $g \in \mathscr{D}(\mathbb{R}_+)$ with $\mathscr{V}$ given by (2.14). In addition, if either

$$\sup_{x \in [0,+\infty)} (Qk'(x) - q'(x)) < +\infty \tag{2.32}$$

or

$$q' \in L^\infty(\mathbb{R}_+) \qquad \text{and} \qquad Qk(x) - q(x) < 0 \quad \text{in a neighbourhood of} \quad x = 0 \tag{2.33}$$

the convergence (2.29) holds true for the whole sequence (f^Δ, u^Δ).

Let us mention here that existence and uniqueness of weak solutions to (2.12), (2.13), (2.15) have been investigated recently under various assumptions on the data k, q and f^{in}[18, 14, 19] and the proof of Theorem 2.2 actually provides an alternative proof of ref. 14, Theorem 2.2 for the class of functions k and q considered here. In addition the last assertion of Theorem 2.2 follows at once from the uniqueness of weak solutions to (2.12), (2.13), (2.15) which is valid when k and q fulfil either (2.32) or (2.33) by ref. 14, Theorem 2.3. Since the functions k and q given by (2.17) and (2.18) enjoy the properties (2.33) and (2.32), respectively, the convergence of the whole sequence (f^Δ, u^Δ) holds true in that case.

Remark 2.3. It is worth to point out here that the scaling of the coagulation coefficients (a_i^Δ) defined in (2.24) and (2.26) is different according to whether $i \geqslant 2$ or $i = 1$. As already remarked in ref. 10 this stems from the fact that, though the 1-clusters play the role of the solute in the Becker–Döring model, the interaction between the solute and the grains in the Lifshitz–Slyozov–Wagner theory and between the 1-clusters and the other clusters in the Becker–Döring model are of a different nature as the formation of 2-clusters by aggregation of 1-clusters does not take place in the former. The rate of this reaction thus should somehow vanish in the limit $\Delta \to 0$.

For the second convergence result of this paper we restrict ourselves to homogeneous functions k and q, that is, the model case

$$k(x) = ax^\lambda \qquad \text{and} \qquad q(x) = bx^\mu, \qquad x \in \mathbb{R}_+ \tag{2.34}$$

where a, b are positive real numbers and $0 \leqslant \mu < \lambda \leqslant 1$. This case includes (2.17) as well as (2.18) when $c = 0$. We also consider a function f^{in} such that

$$\begin{cases} f^{in} \in L^1(\mathbb{R}_+; (1+x)\,dx) \cap W^{1,1}(\mathbb{R}_+), & f^{in} \geqslant 0 \quad \text{a.e. in } \mathbb{R}_+ \text{ and} \\ \int_x^\infty y f^{in}(y)\,dy > 0, & x \in \mathbb{R}_+ \end{cases} \tag{2.35}$$

Remark 2.4. On the one hand the regularity assumption $f^{in} \in W^{1,1}(\mathbb{R}_+)$ made in (2.35) is only used in Lemma 5.4 below and could probably be relaxed. On the other hand the fact that f^{in} is not compactly supported is crucial in order to guarantee a positive lower bound for A^ε defined in (5.11).

We now introduce the Becker–Döring equations approximating the Lifshitz–Slyozov–Wagner equations (2.12), (2.13), (2.16): we define

$$a_i = a i^\lambda \qquad \text{and} \qquad b_{i+1} = b(i+1)^\mu, \qquad i \geqslant 1 \tag{2.36}$$

For $\varepsilon \in (0, 1)$ and $i \geqslant 1$ we put $\Lambda_i = [(i-1/2)\,\varepsilon, (i+1/2)\,\varepsilon)$, $\chi_i^\varepsilon = \mathbf{1}_{\Lambda_i}$ and

$$c_i^{in,\varepsilon} = \varepsilon \int_{\Lambda_i} f^{in}(x)\,dx, \qquad i \geqslant 1 \tag{2.37}$$

$$\alpha_i^\varepsilon = a_i, \qquad \beta_i^\varepsilon = b_i \qquad \text{for} \quad i \geqslant 2 \qquad \text{and} \qquad \alpha_1^\varepsilon = \varepsilon^{3-\lambda} a_1 \tag{2.38}$$

Here again α_1^ε vanishes as $\varepsilon \to 0$ (see Remark 2.3 earlier). Since f^{in} is non-negative by (2.35) and

$$\sum_{i=1}^{\infty} i c_i^{in,\varepsilon} \leqslant 2 \int_0^\infty x f^{in}(x)\,dx \tag{2.39}$$

we have $c^{in,\varepsilon} = (c_i^{in,\varepsilon}) \in X^+$. We then infer from Theorem 2.1 that there exists a unique solution $\Gamma^\varepsilon = (\Gamma_i^\varepsilon)$ to the Becker–Döring equations (1.6), (1.7), (2.1) with kinetic coefficients (α_i^ε), (β_i^ε) and initial data $c^{in,\varepsilon}$ enjoying the properties (2.4)–(2.9). Following the arguments presented in the Introduction we further introduce $c^\varepsilon = (c_i^\varepsilon)$ defined by

$$c_i^\varepsilon(t) = \Gamma_i^\varepsilon(t\varepsilon^{\mu-1}), \qquad (t, i) \in [0, +\infty) \times \mathbb{N} \setminus \{0\} \tag{2.40}$$

and put

$$f^\varepsilon(t, x) = \frac{1}{\varepsilon^2} \sum_{i=2}^{\infty} c_i^\varepsilon(t)\, \chi_i^\varepsilon(x) \qquad \text{and} \qquad u^\varepsilon(t) = \varepsilon^{\mu-\lambda} c_1^\varepsilon(t) \tag{2.41}$$

for $(t, x) \in [0, +\infty) \times \mathbb{R}_+$. Our second result then reads as follows.

Theorem 2.5. There are sequences (ε_n) of real numbers in $(0, 1)$, $\varepsilon_n \to 0$, and a couple of non-negative functions (f, u) such that

$$\begin{cases} f^{\varepsilon_n} \longrightarrow f & \text{in } \mathscr{C}([0, t]; w-L^1(\mathbb{R}_+; x\,dx)) \\ u^{\varepsilon_n} \overset{*}{\rightharpoonup} u & \text{weakly in } L^\infty(0, t) \end{cases} \tag{2.42}$$

Here f is a weak solution to (2.12), (2.13), (2.16) in the following sense:

$$\begin{cases} f \in \mathscr{C}([0, t]; L^1(\mathbb{R}_+; x\,dx)) \cap L^\infty(0, t; L^1(\mathbb{R}_+)) \\ u \in L^\infty(0, t) \end{cases} \tag{2.43}$$

and there holds

$$\int_0^\infty f(t, x)\, g(x)\, dx = \int_0^\infty f^{in}(x)\, g(x)\, dx + \int_0^t \int_0^\infty g_x(x)\, \mathscr{V}(s, x)\, f(s, x)\, dx\, ds \tag{2.44}$$

for each $t \in \mathbb{R}_+$ and $g \in \mathscr{D}(\mathbb{R}_+)$ with $\mathscr{V}$ given by (2.14), k and q by (2.34) and

$$u(t) \int_0^\infty k(x)\, f(t, x)\, dx = \int_0^\infty q(x)\, f(t, x)\, dx \tag{2.45}$$

or, equivalently,

$$\int_0^\infty x f(t, x)\, dx = \int_0^\infty x f^{in}(x)\, dx \tag{2.46}$$

In addition, if either $\mu > 0$ or $\lambda = 1$ the convergence (2.42) holds true for the whole sequence $(f^\varepsilon, u^\varepsilon)$.

The last assertion of Theorem 2.5 readily follows from the uniqueness of weak solutions to (2.12), (2.13), (2.16) which holds true when $\mu > 0$ or $\lambda = 1$ by ref. 13, Theorem 3. Let us also mention here that existence and uniqueness of measure-valued solutions to (2.12), (2.13), (2.16) have been recently proved when k and q are given by (2.17) in refs. 19 and 20 while existence of weak solutions as in the previous theorem is investigated in ref. 13 for a wider class of functions k and q, the initial datum being a non-negative and integrable function with finite first moment.

Remark 2.6. Coming back to the model case (2.34), notice that the case $\lambda = \mu \in [0, 1]$ is neither covered by Theorem 2.5 (as it would formally give the LSW equations (2.12), (2.13), (2.15) in the limit $\varepsilon \to 0$ by (1.12) and not the LSW equations (2.12), (2.13), (2.16)) nor by Theorem 2.2 (since

(2.21) is not fulfilled for U large enough). Such a choice of the functions k and q seems however to be physically irrelevant since, at a given time t, there is no critical radius and all the grains shrink or expand according to the sign of $au(t)-b$. In addition a boundary condition at $x=0$ is needed to solve (1.1) when $ku-q=(u-b/a)\,k$ reaches some positive value. It is thus not clear whether our approach could work in that case.

3. UNIQUENESS FOR THE BD EQUATIONS

As already mentioned, owing to (2.10), the existence of a solution c to (1.6), (1.7), (2.1) satisfying the properties stated in Theorem 2.1 follows from ref. 6, Corollaries 2.3 and 2.6. It thus remains to check the uniqueness part of Theorem 2.1.

Consider a sequence c^{in} in X^+ and let c and $\hat{c}$ be two solutions to (1.6), (1.7), (2.1) in the sense of Theorem 2.1 with initial datum c^{in}. In the following we denote by C any positive constant depending only on K, a_1 and $\|c^{in}\|_X$. For $t \in [0, +\infty)$ and $i \geqslant 1$ we put

$$F_i(t) = \sum_{j=i}^{\infty} c_j(t), \qquad \hat{F}_i(t) = \sum_{j=i}^{\infty} \hat{c}_j(t), \qquad E_i(t) = F_i(t) - \hat{F}_i(t)$$

Clearly $c_i = F_i - F_{i+1}$, $\hat{c}_i = \hat{F}_i - \hat{F}_{i+1}$ and it follows from (2.8) that the sequences $(F_i(t))$ and $(\hat{F}_i(t))$ both belong to $\ell^1(\mathbb{N} \setminus \{0\})$ with

$$\sum_{j=i}^{\infty} F_j(t) = \sum_{j=i}^{\infty} (j-i+1)\, c_j(t) \qquad \text{and} \qquad \sum_{j=i}^{\infty} \hat{F}_j(t) = \sum_{j=i}^{\infty} (j-i+1)\, \hat{c}_j(t) \tag{3.1}$$

for $t \in [0, +\infty)$. Furthermore we deduce from ref. 6, Corollary 2.6 that

$$\frac{dF_i}{dt} = J_{i-1}(c) \qquad \text{and} \qquad \frac{d\hat{F}_i}{dt} = J_{i-1}(\hat{c})$$

for $i \geqslant 2$, whence

$$\frac{dE_i}{dt} = a_{i-1} c_1 (E_{i-1} - E_i) - b_i (E_i - E_{i+1}) + a_{i-1} \hat{c}_{i-1} (c_1 - \hat{c}_1)$$

Multiplying the above inequation by $sign(E_i)$ we end up with

$$\begin{aligned} \frac{d\,|E_i|}{dt} \leqslant{}& a_{i-1} c_1\, |E_{i-1}| + b_i\, |E_{i+1}| - (a_{i-1} c_1 + b_i)\, |E_i| \\ &+ a_{i-1} \hat{c}_{i-1}\, |c_1 - \hat{c}_1| \end{aligned} \tag{3.2}$$

for $i \geqslant 2$. Now let $N \geqslant 3$ and sum the inequality (3.2) for $i \in \{2,\dots, N\}$. We thus obtain

$$\begin{aligned}\frac{d}{dt}\sum_{i=2}^{N}|E_i| &= \sum_{i=1}^{N-1} a_i c_1\,|E_i| + \sum_{i=3}^{N+1} b_{i-1}\,|E_i| \\ &\quad - \sum_{i=2}^{N}(a_{i-1}c_1 + b_i)\,|E_i| + |c_1 - \hat{c}_1|\sum_{i=1}^{N-1} a_i \hat{c}_i \\ &\leqslant a_1 c_1\,|E_1| - a_N c_1\,|E_N| - b_2\,|E_2| + b_N\,|E_{N+1}| \\ &\quad + \sum_{i=2}^{N}(a_i - a_{i-1})\,c_1\,|E_i| + \sum_{i=3}^{N}(b_{i-1} - b_i)\,|E_i| \\ &\quad + \max\{K, a_1\}\,\|\hat{c}^{in}\|_X\,|c_1 - \hat{c}_1|\end{aligned}$$

thanks to (2.8) and (2.10). It then follows from (2.2), (2.3) and (2.8) that

$$\frac{d}{dt}\sum_{i=2}^{N}|E_i| \leqslant a_1\,\|c^{in}\|_X\,|E_1| + b_N\,|E_{N+1}| + K(1+\|c^{in}\|_X)\sum_{i=2}^{N}|E_i| + C\,|c_1 - \hat{c}_1|$$

whence, after integration over $(0, t)$, $t \in \mathbb{R}_+$,

$$\begin{aligned}\sum_{i=2}^{N}|E_i(t)| &\leqslant C\int_0^t\left(\sum_{i=2}^{N}|E_i(s)| + |c_1(s) - \hat{c}_1(s)|\right)ds \\ &\quad + \int_0^t (C\,|E_1(s)| + b_N\,|E_{N+1}(s)|)\,ds \qquad (3.3)\end{aligned}$$

On the one hand, observe that

$$E_1 = c_1 - \hat{c}_1 + E_2 \qquad (3.4)$$

while (2.8) and (3.1) (with $i = 2$) yield that

$$c_1 - \hat{c}_1 = -\sum_{j=2}^{\infty} j c_j + \sum_{j=2}^{\infty} j\hat{c}_j = -E_2 - \sum_{j=2}^{\infty} E_j$$

whence

$$|c_1 - \hat{c}_1| \leqslant 2\sum_{i=2}^{\infty}|E_i| \qquad (3.5)$$

On the other hand we infer from (2.11) that

$$\int_0^t b_N F_{N+1}(s)\, ds \leqslant \int_0^t B_N \sum_{i=N+1}^{\infty} c_i(s)\, ds \leqslant \int_0^t \sum_{i=N+1}^{\infty} B_i c_i(s)\, ds$$

$$\int_0^t \sum_{i=N+1}^{\infty} B_i c_i(s)\, ds \leqslant \int_0^t \sum_{i=N+1}^{\infty} b_i c_i(s)\, ds + K \int_0^t \sum_{i=N+1}^{\infty} i c_i(s)\, ds$$

and (2.5) and (2.8) warrant that the right-hand side of the last inequality converges to zero as $N \to +\infty$. Since a similar result holds for $\hat{c}$ we conclude that

$$\lim_{N \to +\infty} \int_0^t b_N\, |E_{N+1}(s)|\, ds = 0$$

We may then pass to the limit as $N \to +\infty$ in (3.3) and use (3.4) and (3.5) to obtain

$$\sum_{i=2}^{\infty} |E_i(t)| \leqslant C \int_0^t \sum_{i=2}^{\infty} |E_i(s)|\, ds$$

The Gronwall lemma finally yields

$$\sum_{i=2}^{\infty} |E_i(t)| = 0$$

and the proof of the uniqueness statement of Theorem 2.1 is complete. ∎

4. CONVERGENCE TOWARDS (2.12), (2.13), (2.15)

We first notice that, if (f, u) is a solution to the LSW equations (2.12), (2.13), (2.15), then (Af, u) is also a solution with $A = 1$ in (2.15). We may thus take $A = 1$ in (2.15) without loss of generality. We next consider $\Delta \in (0, 1)$ and let (a_i^Δ), (b_i^Δ), c^Δ, f^Δ and u^Δ be defined by (2.24), (2.25), (2.26), (2.27) and (2.28), respectively. We also put

$$a^\Delta = \sum_{i=2}^{\infty} \Delta a_i^\Delta \chi_i^\Delta \qquad \text{and} \qquad b^\Delta = \sum_{i=2}^{\infty} \Delta b_i^\Delta \chi_i^\Delta.$$

It follows from (2.19), (2.20) and (2.21) that

$$\left\{ \begin{array}{l} (a^\Delta, b^\Delta) \text{ are two sequences of non-negative and non-decreasing} \\ \text{functions converging towards } (k, q) \text{ uniformly on compact subsets} \\ \text{of } \mathbb{R}_+ \text{ and there is a constant } C \text{ depending only on } k \text{ and } q \text{ such that} \\ \qquad a^\Delta(x) + b^\Delta(x) \leqslant C(1+x), \qquad x \in \mathbb{R}_+, \end{array} \right. \tag{4.1}$$

$$\begin{cases} \text{for every } U \in \mathbb{R}_+, \text{ there exists } X_U \in (0,1) \text{ depending only on } U, k \\ \text{and } q \text{ such that} \\ \qquad U a^\Delta(x) - b^\Delta(x) + (i-1/2)(b^\Delta(x+\Delta) - b^\Delta(x)) \leqslant 0, \qquad x \in \Lambda_i \\ \text{for } i \geqslant 2 \text{ such that } \Lambda_i \subset (0, X_U) \text{ and } \Delta \in (0, X_U/2) \end{cases} \tag{4.2}$$

While the proof of (4.1) is straightforward we briefly outline that of (4.2) at the end of this section.

The first step of the proof is to identify the equations satisfied by (f^Δ, u^Δ) together with an inequality satisfied by $\beta(f^\Delta)$ for some convex functions β.

Lemma 4.1. Let ξ be a non-negative function in $W^{1,\infty}_{\text{loc}}(\mathbb{R}_+)$ with $\partial_x \xi \in L^\infty(\mathbb{R}_+)$. For $t \geqslant 0$ there holds

$$\int_0^\infty \xi(x)(f^\Delta(t,x) - f^\Delta(0,x))\,dx$$

$$= \mathscr{P}^\Delta(t,\xi) + \int_0^t \int_0^\infty \{(\tau_\Delta \xi)\, a^\Delta u^\Delta - (\tau_{-\Delta}\xi)\, b^\Delta\}\, f^\Delta\,dx\,ds \tag{4.3}$$

$$u^\Delta(t) + \int_0^\infty x f^\Delta(t,x)\,dx = Q + \omega(\Delta) \tag{4.4}$$

where $|\omega(\Delta)| \leqslant 2\,|f^{in}|_{L^1}\,\Delta$,

$$\mathscr{P}^\Delta(t,\xi) = \frac{1}{\Delta^2} \int_0^t \int_{\Lambda_2} (a_1^\Delta u^\Delta(s)\, c_1^\Delta(s)\, \xi(x) - \Delta^2 b^\Delta(x)\, f^\Delta(s,x)\, \xi(x-\Delta))\,dx\,ds$$

and

$$(\tau_h \xi)(x) = \frac{\xi(x+h) - \xi(x)}{h}, \qquad (x,h) \in \mathbb{R}_+ \times \mathbb{R} \setminus \{0\}$$

In addition, if $\beta \in \mathscr{C}^1([0,+\infty))$ is a non-negative and convex piecewise $\mathscr{C}^2$-smooth function with $\beta(0) = 0$, $\beta'(0) \geqslant 0$, β' concave and such that $\xi(.)\,\beta(f^\Delta(0,.))$ belongs to $L^1(\mathbb{R}_+)$, we have

$$\int_0^\infty \xi(x)(\beta(f^\Delta(t,x)) - \beta(f^\Delta(0,x)))\,dx$$

$$\leqslant \Delta a_1^\Delta\, |\xi|_{L^\infty(0,3\Delta)} \int_0^t u^\Delta(s)\, \beta\left(\frac{c_1^\Delta(s)}{\Delta^2}\right) ds$$

$$+ \int_0^t \int_0^\infty \{(\tau_\Delta \xi)\, a^\Delta u^\Delta - (\tau_{-\Delta}\xi)\, b^\Delta + \xi(\tau_\Delta b^\Delta)\}\, \beta(f^\Delta)\,dx\,ds \tag{4.5}$$

Proof. Let us first recall that, by ref. 6, Theorem 2.5, there holds

$$\sum_{i=2}^{\infty} g_i(c_i^{\Delta}(t)-c_i^{in,\Delta}) = \int_0^t \sum_{i=2}^{\infty} (g_{i+1}-g_i)\, a_i^{\Delta} u^{\Delta} c_i^{\Delta}\, ds + \int_0^t g_2 a_1^{\Delta} u^{\Delta} c_1^{\Delta}\, ds$$
$$- \int_0^t \sum_{i=2}^{\infty} (g_i - g_{i-1})\, b_i^{\Delta} c_i^{\Delta}\, ds - \int_0^t g_1 b_2^{\Delta} c_2^{\Delta}\, ds \tag{4.6}$$

for every sequence $(g_i)_{i \geqslant 1}$ of non-negative real numbers such that $(|g_{i+1}-g_i|)$ is bounded. Let ξ be as in the statement of Lemma 4.1 and put

$$g_i = \int_{\Lambda_i} \xi(x)\, dx, \qquad i \geqslant 1$$

Then $(|g_{i+1}-g_i|)$ is bounded and we infer from (4.6) that

$$\sum_{i=2}^{\infty} \int_{\Lambda_i} \xi(x)(c_i^{\Delta}(t)-c_i^{in,\Delta})\, dx$$
$$= \int_0^t \sum_{i=2}^{\infty} \int_{\Lambda_i} (\xi(x+\Delta)-\xi(x))\, a_i^{\Delta} c_1^{\Delta} c_i^{\Delta}\, dx\, ds$$
$$+ \int_0^t \sum_{i=2}^{\infty} \int_{\Lambda_i} (\xi(x-\Delta)-\xi(x))\, b_i^{\Delta} c_i^{\Delta}\, dx\, ds + \Delta^2 \mathscr{P}^{\Delta}(t,\xi)$$

whence (4.3). Next, by (2.8), (2.25) and (2.26) we have

$$u^{\Delta}(t) + \int_0^{\infty} x f^{\Delta}(t,x)\, dx = c_1^{\Delta}(t) + \sum_{i=2}^{\infty} i c_i^{\Delta}(t)$$
$$= c_1^{in,\Delta} + \sum_{i=2}^{\infty} i c_i^{in,\Delta}$$
$$= Q + \omega(\Delta)$$

with

$$\omega(\Delta) = \sum_{i=2}^{\infty} \int_{\Lambda_i} (i\Delta - x)\, f^{in}(x)\, dx - \int_0^{3\Delta/2} x f^{in}(x)\, dx$$

from which we deduce (4.4). We now prove (4.5). For $t \geqslant 0$ we have

$$\int_0^{\infty} \xi(x)(\beta(f^{\Delta}(t,x)) - \beta(f^{\Delta}(0,x)))\, dx$$
$$= \sum_{i=2}^{\infty} g_i \int_0^t \beta'_{\Delta}(c_i^{\Delta}(s))\, \frac{dc_i^{\Delta}}{dt}(s)\, ds$$
$$= \int_0^t (\mathscr{A}(s)\, u^{\Delta}(s) + \mathscr{B}(s))\, ds \tag{4.7}$$

with $\beta_\Delta(r) = \beta(r/\Delta^2)$, $r \geqslant 0$, (g_i) defined as above in terms of ξ and

$$\mathscr{A}(s) = \sum_{i=2}^{\infty} g_i \beta'_\Delta(c_i^\Delta(s))(a_{i-1}^\Delta c_{i-1}^\Delta - a_i^\Delta c_i^\Delta)(s)$$

$$\mathscr{B}(s) = \sum_{i=2}^{\infty} g_i \beta'_\Delta(c_i^\Delta(s))(b_{i+1}^\Delta c_{i+1}^\Delta - b_i^\Delta c_i^\Delta)(s)$$

On the one hand we infer from the convexity of β that

$$\begin{aligned}
\mathscr{A} &= \sum_{i=2}^{\infty} g_i a_{i-1}^\Delta \beta'_\Delta(c_i^\Delta)(c_{i-1}^\Delta - c_i^\Delta) + \sum_{i=2}^{\infty} g_i (a_{i-1}^\Delta - a_i^\Delta)\, \beta'_\Delta(c_i^\Delta)\, c_i^\Delta \\
&\leqslant \sum_{i=2}^{\infty} g_i a_{i-1}^\Delta (\beta_\Delta(c_{i-1}^\Delta) - \beta_\Delta(c_i^\Delta)) + \sum_{i=2}^{\infty} g_i (a_{i-1}^\Delta - a_i^\Delta)\, \beta_\Delta(c_i^\Delta) \\
&\quad + \sum_{i=2}^{\infty} g_i (a_{i-1}^\Delta - a_i^\Delta)(\beta'_\Delta(c_i^\Delta)\, c_i^\Delta - \beta_\Delta(c_i^\Delta)) \\
&\leqslant \sum_{i=2}^{\infty} (g_{i+1} - g_i)\, a_i^\Delta \beta_\Delta(c_i^\Delta) + g_2 a_1^\Delta \beta_\Delta(c_1^\Delta) \\
&\quad + \sum_{i=2}^{\infty} g_i (a_{i-1}^\Delta - a_i^\Delta)(\beta'_\Delta(c_i^\Delta)\, c_i^\Delta - \beta_\Delta(c_i^\Delta))
\end{aligned}$$

$$\mathscr{A} \leqslant \Delta\, |\xi|_{L^\infty(0,3\Delta)}\, a_1^\Delta \beta_\Delta(c_1^\Delta) + \sum_{i=2}^{\infty} (g_{i+1} - g_i)\, a_i^\Delta \beta_\Delta(c_i^\Delta)$$

where the last inequality follows from the monotonicity (2.19) of k and the property $r\beta'_\Delta(r) - \beta_\Delta(r) \geqslant 0$ for $r \geqslant 0$ enjoyed by non-negative convex functions vanishing at $r = 0$. Writing g_i in terms of ξ yields

$$\mathscr{A} \leqslant \Delta a_1^\Delta\, |\xi|_{L^\infty(0,3\Delta)}\, \beta(c_1^\Delta/\Delta^2) + \int_0^\infty (\tau_\Delta \xi)\, a^\Delta \beta(f^\Delta)\, dx$$

On the other hand we use once more the convexity of β to obtain

$$\begin{aligned}
\mathscr{B} &= \sum_{i=2}^{\infty} g_i b_{i+1}^\Delta \beta'_\Delta(c_i^\Delta)(c_{i+1}^\Delta - c_i^\Delta) + \sum_{i=2}^{\infty} g_i (b_{i+1}^\Delta - b_i^\Delta)\, \beta'_\Delta(c_i^\Delta)\, c_i^\Delta \\
&\leqslant \sum_{i=2}^{\infty} g_i b_{i+1}^\Delta (\beta_\Delta(c_{i+1}^\Delta) - \beta_\Delta(c_i^\Delta)) + \sum_{i=2}^{\infty} g_i (b_{i+1}^\Delta - b_i^\Delta)\, \beta_\Delta(c_i^\Delta) \\
&\quad + \sum_{i=2}^{\infty} g_i (b_{i+1}^\Delta - b_i^\Delta)(\beta'_\Delta(c_i^\Delta)\, c_i^\Delta - \beta_\Delta(c_i^\Delta))
\end{aligned}$$

$$\leqslant \sum_{i=2}^{\infty} (g_{i-1} - g_i)\, b_i^\Delta \beta_\Delta(c_i^\Delta) + \sum_{i=2}^{\infty} g_i (b_{i+1}^\Delta - b_i^\Delta)(\beta'_\Delta(c_i^\Delta)\, c_i^\Delta - \beta_\Delta(c_i^\Delta))$$

$$\mathscr{B} \leqslant \sum_{i=2}^{\infty} (g_{i-1} - g_i)\, b_i^\Delta \beta_\Delta(c_i^\Delta) + \sum_{i=2}^{\infty} g_i (b_{i+1}^\Delta - b_i^\Delta)\, \beta_\Delta(c_i^\Delta)$$

where we have used the monotonicity (2.20) of q and the inequality $r\beta'_\Delta(r) \leqslant 2\beta_\Delta(r)$ which holds true for β_Δ by ref. 14, Lemma A.1. We thus end up with

$$\mathscr{B} \leqslant \int_0^\infty \{\xi(\tau_\Delta b^\Delta) - (\tau_{-\Delta}\xi)\, b^\Delta\}\, \beta(f^\Delta)\, dx$$

Inserting the estimates for $\mathscr{A}$ and $\mathscr{B}$ in (4.7) yields (4.5). ∎

As a first consequence of (4.4) and the non-negativity of u^Δ and f^Δ there are positive constants U and C depending only on Q and f^{in} such that

$$u^\Delta(t) \leqslant U \qquad \text{and} \qquad \int_0^\infty x f^\Delta(t, x)\, dx \leqslant C \tag{4.8}$$

for every $(t, \Delta) \in [0, +\infty) \times (0, 1)$. In the following we denote by C any positive constant which depends only on Q, k, q and f^{in}. The dependence of C upon additional parameters will be indicated explicitly.

We fix $T \in \mathbb{R}_+$ and assume from now on that $\Delta \in (0, X_U/4)$ where X_U is defined in (4.2). It follows from (4.3) with $\xi \equiv 1$ that

$$\|f^\Delta(t)\|_{L^1} \leqslant \|f^\Delta(0)\|_{L^1} + \frac{a_1^\Delta}{\Delta^2} \int_0^t u^\Delta(s)\, c_1^\Delta(s)\, ds, \qquad t \in [0, T]$$

and we deduce from (2.25), (2.26) and (4.8) that

$$\|f^\Delta(t)\|_{L^1} \leqslant \|f^{in}\|_{L^1} + \|k\|_{L^\infty(0,2)}\, U^2 T \leqslant C(T), \qquad t \in [0, T] \tag{4.9}$$

We next investigate the propagation of generalized moments of f^Δ.

Lemma 4.2. Let $\varphi \in \mathscr{C}^1([0, +\infty))$ be a non-negative and convex piecewise $\mathscr{C}^2$-smooth function with $\varphi(0) = 0$, $\varphi'(0) \geqslant 0$ and such that φ' is a concave function. Assume further that

$$M_\varphi := \int_0^\infty \varphi(x)\, f^{in}(x)\, dx < \infty \tag{4.10}$$

For $t \in [0, T]$ there holds

$$\int_0^\infty \varphi(x)\, f^\Delta(t, x)\, dx \leqslant C(T, \varphi''(0), M_\varphi) \tag{4.11}$$

Proof. We first assume that φ' is bounded. We may then take $\xi = \varphi$ in (4.3): since the convexity of φ and the concavity of φ' entail that

$$\begin{aligned}(\tau_\Delta\varphi - \tau_{-\Delta}\varphi)(x) &\leqslant \varphi'(x+\Delta) - \varphi'(x-\Delta) \\ &\leqslant 2\Delta\varphi''(x-\Delta) \leqslant 2\Delta\varphi''(0)\end{aligned}$$

for $x \geqslant \Delta$, we infer from (2.26), (4.8) and the monotonicity of φ that, for $t \in [0, T]$,

$$\begin{aligned}\int_0^\infty \varphi(f^\Delta(t) - f^\Delta(0))\, dx &\leqslant C(T) + 2\Delta\varphi''(0) \int_0^t \int_0^\infty b^\Delta f^\Delta\, dx\, ds \\ &\quad + \int_0^t \int_0^\infty (\tau_\Delta\varphi)(a^\Delta U - b^\Delta)\, f^\Delta\, dx\, ds\end{aligned}$$

We use again the monotonicity of φ together with (4.1), (4.2), (4.8) and (4.9) to conclude that

$$\begin{aligned}\int_0^t \int_0^\infty (\tau_\Delta\varphi)(a^\Delta U - b^\Delta)\, f^\Delta\, dx\, ds &\leqslant \int_0^t \int_0^{X_U} (\tau_\Delta\varphi)(a^\Delta U - b^\Delta)\, f^\Delta\, dx\, ds \\ &\quad + \int_0^t \int_{X_U}^\infty (\tau_\Delta\varphi)(a^\Delta U - b^\Delta)\, f^\Delta\, dx\, ds \\ &\leqslant C \int_0^t \int_{X_U}^\infty (1+x)(\tau_\Delta\varphi)\, f^\Delta\, dx\, ds \\ &\leqslant C \int_0^t \int_{X_U}^\infty x(\tau_\Delta\varphi)\, f^\Delta\, dx\, ds\end{aligned}$$

and

$$2\Delta\varphi''(0) \int_0^t \int_0^\infty b^\Delta f^\Delta\, dx\, ds \leqslant C(T, \varphi''(0))$$

Consequently there holds

$$\int_0^\infty \varphi(f^\Delta(t) - f^\Delta(0))\, dx \leqslant C(T, \varphi''(0)) \left(1 + \int_0^t \int_{X_U}^\infty x(\tau_\Delta\varphi)\, f^\Delta\, dx\, ds\right) \tag{4.12}$$

We now infer from the properties of φ and ref. 14, Lemma A.1 that $x\varphi'(x) \leqslant 2\varphi(x)$ for $x \geqslant 0$ while the concavity of φ' implies that $\varphi'(2x) \leqslant 2\varphi'(x)$, whence $\varphi(2x) \leqslant 4\varphi(x)$, $x \geqslant 0$. Consequently, for $x \geqslant X_U \geqslant \Delta$,

$$x(\tau_\Delta \varphi)(x) \leqslant \frac{1}{\Delta} \int_x^{x+\Delta} y\varphi'(y)\, dy \leqslant \frac{2}{\Delta} \int_x^{x+\Delta} \varphi(y)\, dy \leqslant 2\varphi(2x) \leqslant 8\varphi(x)$$

Inserting the above estimate in (4.12) we obtain

$$\int_0^\infty \varphi(f^\Delta(t) - f^\Delta(0))\, dx \leqslant C(T, \varphi''(0)) \left(1 + \int_0^t \int_0^\infty \varphi f^\Delta\, dx\, ds\right) \tag{4.13}$$

Finally, since φ is non-decreasing, it follows from (2.25) that

$$\int_0^\infty \varphi f^\Delta(0)\, dx \leqslant \sum_{i=2}^\infty \int_{\Lambda_i} \varphi((i+1/2)\, \Delta)\, f^{in}\, dx$$

$$\leqslant \sum_{i=2}^\infty \int_{\Lambda_i} \varphi(2x)\, f^{in}(x)\, dx \leqslant 4 \int_0^\infty \varphi f^{in}\, dx$$

Recalling (4.13) we have thus shown that

$$\int_0^\infty \varphi f^\Delta(t)\, dx \leqslant C(T, \varphi''(0), M_\varphi) \left(1 + \int_0^t \int_0^\infty \varphi f^\Delta\, dx\, ds\right)$$

for $t \in [0, T]$, whence (4.11) by the Gronwall lemma provided φ' is bounded. In the general case we introduce φ_R defined by

$$\varphi_R(y) = \begin{cases} \varphi(y) & \text{if } \ y \in [0, R] \\ \varphi'(R)(y - R) + \varphi(R) & \text{if } \ y \in [R, +\infty) \end{cases}$$

for $R \geqslant 2$ and notice that φ_R enjoys the same properties as φ with a bounded first derivative and $\varphi_R \leqslant \varphi$ with $\varphi_R''(0) = \varphi''(0)$. The previous computation may then be performed with φ_R and Lemma 4.2 follows by passing to the limit as $R \to +\infty$ after noticing that the constant in (4.11) does not depend on R. ∎

We next employ (4.5) to study the behaviour of some superlinear functionals of f^Δ.

Lemma 4.3. Let $\beta \in \mathscr{C}^1([0, +\infty))$ be a non-negative and convex piecewise $\mathscr{C}^2$-smooth function with $\beta(0) = 0$, $\beta'(0) = 0$ and such that β' is a concave function. Assume further that

$$L_\beta := \int_0^\infty \beta(f^{in}(x))\, dx < \infty \tag{4.14}$$

For $t \in [0, T]$ there holds

$$\int_0^\infty \min\{x, 1\}\, \beta(f^\Delta(t, x))\, dx \leqslant C(T, \beta, L_\beta) \tag{4.15}$$

Proof. Since $\Delta \leqslant X_U/4$ there is an integer $i_\star \geqslant 4$ such that $X_U \in \Lambda_{i_\star}$. We then define

$$\xi = \Delta \sum_{i=2}^{\infty} (\min(i, i_\star) - 1/2)\, \chi_i^\Delta$$

Owing to the specific structure of ξ we can still use (4.5). Since $\xi(x) \leqslant x$ it follows from (4.5) and (4.8) that

$$\begin{aligned}
&\int_0^\infty \xi(\beta(f^\Delta(t)) - \beta(f^\Delta(0)))\, dx \\
&\quad \leqslant 3\, \Delta^2 a_1^\Delta T U \beta(U/\Delta^2) \\
&\qquad + \int_0^t \int_{3\Delta/2}^{(i_\star - 1/2)\,\Delta} ((\tau_\Delta \xi)\, a^\Delta U - (\tau_{-\Delta}\xi)\, b^\Delta + \xi(\tau_\Delta b^\Delta))\, \beta(f^\Delta)\, dx\, ds \\
&\qquad + \int_0^t \int_{(i_\star - 1/2)\,\Delta}^\infty ((\tau_\Delta \xi)\, a^\Delta U - (\tau_{-\Delta}\xi)\, b^\Delta + \xi(\tau_\Delta b^\Delta))\, \beta(f^\Delta)\, dx\, ds
\end{aligned}$$

Now, if $x \in (3\Delta/2, (i_\star - 1/2)\,\Delta)$ there is $i \in \{2, ..., i_\star - 1\}$ such that $x \in \Lambda_i$. Since $\Lambda_i \subset (0, X_U)$, (4.2) ensures that

$$\begin{aligned}
&((\tau_\Delta \xi)\, a^\Delta U - (\tau_{-\Delta}\xi)\, b^\Delta + \xi(\tau_\Delta b^\Delta))(x) \\
&\quad \leqslant a^\Delta(x)\, U - b^\Delta(x) + (i - 1/2)\, \Delta(\tau_\Delta b^\Delta)(x) \leqslant 0
\end{aligned}$$

while, for $x \geqslant (i_\star - 1/2)\,\Delta$, $x \in \Lambda_i$ for some $i \geqslant i_\star$ and there holds

$$\begin{aligned}
&((\tau_\Delta \xi)\, a^\Delta U - (\tau_{-\Delta}\xi)\, b^\Delta + \xi(\tau_\Delta b^\Delta))(x) \\
&\quad \leqslant (i_\star - 1/2)\, \Delta \int_{\Lambda_i} \frac{q(y+\Delta) - q(y)}{\Delta}\, dy \leqslant |q'|_{L^\infty(X_U/2, +\infty)}\, \xi(x)
\end{aligned}$$

Consequently,

$$\begin{aligned}
\int_0^\infty \xi \beta(f^\Delta(t))\, dx &\leqslant \int_0^\infty \xi \beta(f^\Delta(0))\, dx + C(T)\, \Delta^2 a_1^\Delta \beta(U/\Delta^2) \\
&\quad + C \int_0^t \int_0^\infty \xi \beta(f^\Delta(s))\, dx\, ds
\end{aligned}$$

Finally, thanks to (2.26), the Jensen inequality and the properties of β, we have

$$\int_0^\infty \xi\beta(f^\Delta(0))\,dx \leqslant C\sum_{i=2}^{\infty}\int_{\Lambda_i}\beta(f^{in}(x))\,dx \leqslant L_\beta$$

and

$$\Delta^2 a_1^\Delta \beta(U/\Delta^2) \leqslant C\Delta^4\left(\beta'(0)\,\frac{U}{\Delta^2}+\beta''(0)\,\frac{U^2}{\Delta^4}\right) \leqslant C(\beta)$$

from which we deduce that

$$\int_0^\infty \xi\beta(f^\Delta(t))\,dx \leqslant C(T,\beta,L_\beta)\left(1+\int_0^t\int_0^\infty \xi\beta(f^\Delta(s))\,dx\,ds\right)$$

for $t\in[0,T]$. Since $\xi(x)\geqslant \min\{x,1\}/3$ for $x\geqslant 3\Delta/2$ the Gronwall lemma allows us to conclude that (4.15) holds true. ▮

We are now in a position to complete the proof of Theorem 2.2 and first aim at showing that Lemmas 4.2 and 4.3 provide useful information on the compactness properties of f^Δ. We previously recall that (2.23) and a refined version of the de la Vallée-Poussin theorem in ref. 21, Proposition I.1.1 warrant that there are two non-negative and convex functions Φ_1 and Φ_2 such that, for $l=1,2$, Φ_l belongs to $\mathscr{C}^1([0,+\infty))$ and is piecewise $\mathscr{C}^2$-smooth with $\Phi_l(0)=0$, $\Phi_l'(0)\geqslant 0$, Φ_l' is a concave function,

$$\lim_{r\to+\infty}\frac{\Phi_l(r)}{r}=+\infty \tag{4.16}$$

and

$$L:=\int_0^\infty \{\Phi_1(1+x)\,f^{in}(x)+(1+x)\,\Phi_2(f^{in}(x))\}\,dx<+\infty \tag{4.17}$$

Owing to (4.17) we deduce from Lemmas 4.2 and 4.3 that

$$\sup_{t\in[0,T]}\int_0^\infty \{\Phi_1(x)\,f^\Delta(t,x)+\min\{x,1\}\,\Phi_2(f^\Delta(t,x))\}\,dx \leqslant C(T) \tag{4.18}$$

Since Φ_1 and Φ_2 satisfy (4.16), the bound (4.18) and the Dunford–Pettis theorem entail that there is a weakly compact subset $\mathscr{K}_w(T)$ of $L^1(\mathbb{R}_+, x\,dx)$ such that

$$\{f^\Delta(t),\,\Delta\in(0,X_U/4)\}\subset\mathscr{K}_w(T) \qquad \text{for} \quad t\in[0,T] \tag{4.19}$$

We next investigate the equicontinuity with respect to time of (f^{Δ}). Fix $R\geqslant 1$ and consider $\xi\in\mathscr{D}(1/R, R)$. If $\Delta\leqslant 1/(5R)$ it follows from (4.1), (4.3), (4.8) and (4.9) that, for $t\in[0,T)$ and $h\in(0,T-t)$, we have

$$\left|\int_0^\infty \xi(x)(f^{\Delta}(t+h,x)-f^{\Delta}(t,x))\,dx\right|$$

$$\leqslant \int_t^{t+h} |\partial_x\xi|_{L^\infty}\int_0^\infty (a^{\Delta}U+b^{\Delta})\,f^{\Delta}\,dx\,ds$$

$$\leqslant C(\xi)\int_t^{t+h}\int_0^\infty (1+x)\,f^{\Delta}\,dx\,ds\leqslant C(\xi,T)\,h$$

whence

$$\lim_{h\to 0}\ \sup_{t\in[0,T-h]}\ \sup_{\Delta\in(0,1/(5R))}\left|\int \xi(x)(f^{\Delta}(t+h,x)-f^{\Delta}(t,x))\,dx\right|=0 \tag{4.20}$$

Furthermore, since an arbitrary function in $L^\infty(1/R, R)$ is the almost everywhere limit of a sequence of functions in $\mathscr{D}(1/R, R)$ which is bounded in $L^\infty(1/R, R)$ we infer from (4.19) and (4.20) that (4.20) actually holds true for every $\xi\in L^\infty(1/R, R)$. According to a variant of the Arzelà–Ascoli theorem (see, e.g., ref. 22, Theorem 1.3.2) we infer from (4.19) and (4.20) that (f^{Δ}) is relatively compact in $\mathscr{C}([0,T]; w-L^1(1/R, R))$ for each $T\in\mathbb{R}_+$ and $R\geqslant 1$. By a diagonal process we obtain a sequence (Δ_n) of real numbers in (0, 1), $\Delta_n\to 0$, and a function

$$f\in\mathscr{C}([0,+\infty); w-L^1(1/R, R)) \tag{4.21}$$

such that

$$f^{\Delta_n}\longrightarrow f \qquad \text{in}\quad \mathscr{C}([0,T]; w-L^1(1/R, R)) \tag{4.22}$$

for each $T\in\mathbb{R}_+$ and $R\geqslant 1$. Recalling (4.19) it is easily seen that (4.21)–(4.22) can be improved to $f\in\mathscr{C}([0,+\infty); w-L^1(\mathbb{R}_+, x\,dx))$ and

$$f^{\Delta_n}\longrightarrow f \qquad \text{in}\quad \mathscr{C}([0,T]; w-L^1(\mathbb{R}_+, x\,dx)) \tag{4.23}$$

for each $T\in\mathbb{R}_+$. Clearly (4.23) ensures that $f(t)$ is non-negative almost everywhere in $\mathbb{R}_+$ for $t\geqslant 0$ and it readily follows from (4.4), (4.9) and (4.23) that $f\in L^\infty(0,T; L^1(\mathbb{R}_+))$ and

$$\lim_{n\to+\infty}|u^{\Delta_n}-u|_{\mathscr{C}([0,T])}=0 \qquad \text{with}\quad u(t)=Q-\int_0^\infty xf(t,x)\,dx \tag{4.24}$$

for $t \in [0, T]$ and $T \in \mathbb{R}_+$. In addition the non-negativity of u^Δ and (4.24) yield the non-negativity of u. Finally, owing to (4.1), (4.23), (4.24) and since

$$(\tau_\Delta \xi, \tau_{-\Delta} \xi) \longrightarrow (\partial_x \xi, -\partial_x \xi) \qquad \text{in} \quad L^\infty(0, +\infty) \tag{4.25}$$

for $\xi \in \mathscr{D}(\mathbb{R}_+)$ we may pass to the limit as $\Delta_n \to 0$ in (4.3) and conclude that f satisfies (2.31). The strong continuity of f claimed in (2.30) then follows from (2.31) by arguments similar to the ones developed in ref. 23 and the proof of Theorem 2.2 is complete.

We now sketch the proof of (4.2). Let U be a positive real number and put $X_U = x_U/2$ where x_U is given by (2.21). Fix $\Delta \in (0, X_U/2)$ and notice that $3\Delta/2 \leqslant X_U$. For $i \geqslant 2$ such that $\Lambda_i \subset (0, X_U)$ and $x \in \Lambda_i$ there holds

$$\begin{aligned} & Ua^\Delta(x) - b^\Delta(x) + (i-1/2)(b^\Delta(x+\Delta) - b^\Delta(x)) \\ &\quad = \frac{1}{\Delta}\left(\int_{\Lambda_i} \{Uk(y) - q(y) + (i-1/2)(q(y+\Delta) - q(y))\}\, dy\right) \\ &\quad \leqslant \frac{1}{\Delta}\left(\int_{\Lambda_i} \{Uk(y) - q(y) + (i-1/2)\, \Delta q'(y)\}\, dy\right) \end{aligned}$$

where we have used the concavity of q to obtain the last inequality. Since q' is non-negative and $\Lambda_i \subset (0, x_U)$ we further deduce from the above inequality and (2.21) that

$$\begin{aligned} & Ua^\Delta(x) - b^\Delta(x) + (i-1/2)(b^\Delta(x+\Delta) - b^\Delta(x)) \\ &\quad \leqslant \frac{1}{\Delta}\left(\int_{\Lambda_i} \{Uk(y) - q(y) + yq'(y)\}\, dy\right) \leqslant 0 \end{aligned}$$

whence (4.2).

5. CONVERGENCE TOWARDS (2.12), (2.13), (2.16)

In this section we consider the Lifshitz–Slyozov–Wagner limit of the Becker–Döring equations for homogeneous coefficients. For convenience we first recall the notations introduced in Section 2. Let a, b, λ and μ be positive real numbers such that $0 \leqslant \mu < \lambda \leqslant 1$ and consider an initial datum f^{in} satisfying (2.35). For $\varepsilon \in (0, 1)$ recall that $c^{in, \varepsilon}$ is defined by (2.37) as follows

$$c_i^{in, \varepsilon} = \varepsilon \int_{\Lambda_i} f^{in}(x)\, dx, \qquad i \geqslant 1$$

while the kinetic coefficients (α_i^ε) and (β_i^ε) are defined by (2.38) as

$$\alpha_i^\varepsilon = a i^\lambda, \qquad \beta_i^\varepsilon = b i^\mu \qquad \text{for} \quad i \geqslant 2 \qquad \text{and} \qquad \alpha_1^\varepsilon = \varepsilon^{3-\lambda} a_1$$

Recalling that $\Gamma^\varepsilon = (\Gamma_i^\varepsilon)$ is the solution to the Becker–Döring equations (1.6), (1.7), (2.1) with kinetic coefficients (α_i^ε) , (β_i^ε) and initial data $c^{in, \varepsilon}$ we infer from (1.7) that $c^\varepsilon = (c_i^\varepsilon)$ defined by (2.40), that is,

$$c_i^\varepsilon(t) = \Gamma_i^\varepsilon(t\varepsilon^{\mu-1}), \qquad (t, i) \in [0, +\infty) \times \mathbb{N} \setminus \{0\}$$

satisfies

$$\frac{dc_i^\varepsilon}{dt} = u^\varepsilon(a_{i-1}^\varepsilon c_{i-1}^\varepsilon - a_i^\varepsilon c_i^\varepsilon) + (b_{i+1}^\varepsilon c_{i+1}^\varepsilon - b_i^\varepsilon c_i^\varepsilon) \tag{5.1}$$

for $i \geqslant 2$ where

$$a_i^\varepsilon = \varepsilon^{\lambda-1} \alpha_i^\varepsilon, \qquad b_{i+1}^\varepsilon = \varepsilon^{\mu-1} \beta_{i+1}^\varepsilon, \qquad i \geqslant 1 \tag{5.2}$$

and u^ε is given by (2.41), that is, $u^\varepsilon = \varepsilon^{\mu-\lambda} c_1^\varepsilon$. Therefore $(c_i^\varepsilon)_{i \geqslant 2}$ satisfies a system of equations which is similar to the one satisfied by $(c_i^\Delta)_{i \geqslant 2}$ in the previous section (recall (1.13)) with the difference that c_1^Δ is replaced by u^ε. Arguing as in the previous section we obtain an analogue of Lemma 4.1.

Lemma 5.1. Let ξ be a non-negative function in $W^{1,\infty}_{\text{loc}}(\mathbb{R}_+)$ with $\partial_x \xi \in L^\infty(\mathbb{R}_+)$. For $t \geqslant 0$ there holds

$$\int_0^\infty \xi(x)(f^\varepsilon(t, x) - f^\varepsilon(0, x))\, dx$$

$$= \mathscr{P}^\varepsilon(t, \xi) + \int_0^t \int_0^\infty \{(\tau_\varepsilon \xi)\, a^\varepsilon u^\varepsilon - (\tau_{-\varepsilon} \xi)\, b^\varepsilon\}\, f^\varepsilon\, dx\, ds \tag{5.3}$$

$$\varepsilon^{\lambda-\mu} u^\varepsilon(t) + \int_0^\infty x f^\varepsilon(t, x)\, dx = \int_0^\infty x f^{in}(x)\, dx + \omega(\varepsilon) \tag{5.4}$$

where $|\omega(\varepsilon)| \leqslant 2\, |f^{in}|_{L^1} \varepsilon$,

$$\mathscr{P}^\varepsilon(t, \xi) = \frac{1}{\varepsilon^2} \int_0^t \int_{\Lambda_2} (a_1^\varepsilon u^\varepsilon(s)\, c_1^\varepsilon(s)\, \xi(x) - \varepsilon^2 b^\varepsilon(x)\, f^\varepsilon(s, x)\, \xi(x - \varepsilon))\, dx\, ds$$

$$a^\varepsilon = \sum_{i=2}^\infty \varepsilon a_i^\varepsilon \chi_i^\varepsilon, \qquad b^\varepsilon = \sum_{i=2}^\infty \varepsilon b_i^\varepsilon \chi_i^\varepsilon \tag{5.5}$$

and

$$(\tau_h \xi)(x) = \frac{\xi(x+h) - \xi(x)}{h}, \qquad (x, h) \in \mathbb{R}_+ \times \mathbb{R} \setminus \{0\}$$

In addition, if $\beta \in \mathscr{C}^1([0, +\infty))$ is a non-negative and convex piecewise $\mathscr{C}^2$-smooth function with $\beta(0) = 0$, $\beta'(0) \geqslant 0$, β' concave and such that $\xi(.)\, \beta(f^\varepsilon(0, .))$ belongs to $L^1(\mathbb{R}_+)$, we have

$$\begin{aligned} &\int_0^\infty \xi(x)(\beta(f^\varepsilon(t, x)) - \beta(f^\varepsilon(0, x)))\, dx \\ &\quad \leqslant \varepsilon a_1^\varepsilon\, |\xi|_{L^\infty(0, 3\varepsilon)} \int_0^t u^\varepsilon(s)\, \beta\left(\frac{c_1^\varepsilon(s)}{\varepsilon^2}\right) ds \\ &\qquad + \int_0^t \int_0^\infty \{(\tau_\varepsilon \xi)\, a^\varepsilon u^\varepsilon - (\tau_{-\varepsilon} \xi)\, b^\varepsilon + \xi(\tau_\varepsilon b^\varepsilon)\}\, \beta(f^\varepsilon)\, dx\, ds \end{aligned} \tag{5.6}$$

Owing to Lemma 5.1 we expect to proceed as in the proof of Theorem 2.2 as soon as we are able to show that u^ε is bounded uniformly with respect to $\varepsilon \in (0, 1)$. In contrast to the previous section, the equality (5.4) does not provide the boundedness of u^ε but only that of $c_1^\varepsilon = \varepsilon^{\lambda - \mu} u^\varepsilon$. The boundedness of u^ε is actually the only difficulty to be overcome for the method developed in the previous section to be applied. Before proceeding with this step some preliminary computations are needed which we performed now.

In the following we denote by C any positive constant depending only on a, b, λ, μ and f^{in}. The dependence of C upon additional parameters will be indicated explicitly. Notice first that the functions (a^ε) and (b^ε) defined in (5.5) are indeed approximations of $k(x) = ax^\lambda$ and $q(x) = bx^\mu$, $x \in \mathbb{R}_+$, since

$$\begin{cases} a^\varepsilon \text{ and } b^\varepsilon \text{ converge uniformly on compact subsets of } \mathbb{R}_+ \\ \text{towards } x \mapsto ax^\lambda \text{ and } x \mapsto bx^\mu \text{, respectively, and there holds} \\ \qquad a^\varepsilon(x) + b^\varepsilon(x) \leqslant C(1 + x), \qquad x \in \mathbb{R}_+ \end{cases} \tag{5.7}$$

Also (5.3) and (5.4) allow us to obtain some bounds on (f^ε). Indeed it follows at once from (5.4) that

$$u^\varepsilon(t) \leqslant C\varepsilon^{\mu - \lambda} \qquad \text{and} \qquad \int_0^\infty x f^\varepsilon(t, x)\, dx \leqslant C, \qquad t \in [0, +\infty) \tag{5.8}$$

while (5.3) with $\xi = 1$, (2.38) and (5.2) yield for $T \in \mathbb{R}_+$ and $t \in [0, T]$,

$$|f^\varepsilon(t)|_{L^1} \leqslant |f^{in}|_{L^1} + C\varepsilon^{1+\mu-\lambda} t \leqslant C(T) \tag{5.9}$$

We next derive a differential inequality for u^ε.

Lemma 5.2. For $t \in [0, +\infty)$ there holds

$$\varepsilon^{\lambda-\mu} \frac{du^\varepsilon}{dt}(t) + A^\varepsilon(t)\, u^\varepsilon(t) \leqslant 2B^\varepsilon(t) \tag{5.10}$$

where

$$A^\varepsilon(t) = \int_0^\infty a^\varepsilon(x)\, f^\varepsilon(t, x)\, dx \qquad \text{and} \qquad B^\varepsilon(t) = \int_0^\infty b^\varepsilon(x)\, f^\varepsilon(t, x)\, dx \tag{5.11}$$

Proof. It follows from (5.1) and (5.4) by an argument similar to that of ref. 6, Theorem 2.5 that

$$\begin{aligned}
\varepsilon^{\lambda-\mu} \frac{du^\varepsilon}{dt} &= -\sum_{i=2}^{\infty} i\, \frac{dc_i^\varepsilon}{dt} \\
&= -a_1^\varepsilon u^\varepsilon c_1^\varepsilon + b_2^\varepsilon c_2^\varepsilon - \sum_{i=1}^{\infty} a_i^\varepsilon u^\varepsilon c_i^\varepsilon + \sum_{i=2}^{\infty} b_i^\varepsilon c_i^\varepsilon
\end{aligned}$$

Since the definition (5.5) of a^ε and b^ε yields

$$\sum_{i=2}^{\infty} a_i^\varepsilon c_i^\varepsilon = \sum_{i=2}^{\infty} \int_{\Lambda_i} a^\varepsilon(x)\, f^\varepsilon(., x)\, dx = \int_0^\infty a^\varepsilon(x)\, f^\varepsilon(., x)\, dx$$

$$\sum_{i=2}^{\infty} b_i^\varepsilon c_i^\varepsilon = \int_0^\infty b^\varepsilon(x)\, f^\varepsilon(., x)\, dx$$

we finally obtain

$$\begin{aligned}
\varepsilon^{\lambda-\mu} \frac{du^\varepsilon}{dt} = &-2\varepsilon^{\lambda-\mu} a_1^\varepsilon (u^\varepsilon)^2 + \int_{\Lambda_2} b^\varepsilon(x)\, f^\varepsilon(., x)\, dx \\
&+ \int_0^\infty b^\varepsilon(x)\, f^\varepsilon(., x)\, dx - u^\varepsilon \int_0^\infty a^\varepsilon(x)\, f^\varepsilon(., x)\, dx
\end{aligned}$$

from which (5.10) readily follows. ∎

In order to exploit the differential inequality (5.10) satisfied by u^ε, a positive bound from below on A^ε and a bound from above for B^ε seem to be needed. While the latter is an easy consequence of (5.7), (5.8) and (5.9) which imply that

$$B^\varepsilon(t) \leqslant C(T), \qquad (t, \varepsilon) \in [0, T] \times (0, 1) \tag{5.12}$$

the former seems to be less straightforward to obtain and is achieved by a further development of a device from ref. 13. Introducing

$$F^\varepsilon(t, x) = \int_x^\infty f^\varepsilon(t, y)\, dy, \qquad (t, x) \in [0, +\infty) \times \mathbb{R}_+ \tag{5.13}$$

we notice that

$$A^\varepsilon(t) \geqslant a \int_0^\infty (y-\varepsilon)^\lambda f^\varepsilon(t, y)\, dy \geqslant a(x-\varepsilon)^\lambda F^\varepsilon(t, x) \tag{5.14}$$

for $x \in \mathbb{R}_+$. We thus realize that a lower bound for A^ε may be deduced from a lower bound for F^ε which will be obtained in two steps. We first derive a differential inequality for F^ε.

Lemma 5.3. There is a positive constant D depending only on a, b, λ, μ and f^{in} such that, for $t \in [0, +\infty)$ and $x \in [3\varepsilon, +\infty)$, there holds

$$\partial_t F^\varepsilon(t, x) \geqslant -\frac{D(1+x)}{\varepsilon} (F^\varepsilon(t, x) - F^\varepsilon(t, x+\varepsilon)) \tag{5.15}$$

Proof. We fix $x \in [3\varepsilon, +\infty)$ and consider $t \in [0, +\infty)$ and $s \in (0, t)$. Observing that (5.3) is still valid for $\xi = \mathbf{1}_{(x, +\infty)}$ we obtain

$$\begin{aligned} F^\varepsilon(t, x) - F^\varepsilon(s, x) &\geqslant -\int_s^t \int_0^\infty (\tau_{-\varepsilon} \mathbf{1}_{(x, +\infty)})(y)\, b^\varepsilon(y)\, f^\varepsilon(\sigma, y)\, dy\, d\sigma \\ &\geqslant -\frac{1}{\varepsilon} \int_s^t \int_x^{x+\varepsilon} b^\varepsilon(y)\, f^\varepsilon(\sigma, y)\, dy\, d\sigma \end{aligned}$$

since $\mathbf{1}_{(x, +\infty)}$ is non-decreasing. We next infer from (5.7) that

$$\begin{aligned} F^\varepsilon(t, x) - F^\varepsilon(s, x) &\geqslant -\frac{C}{\varepsilon} \int_s^t \int_x^{x+\varepsilon} (1+y)\, f^\varepsilon(\sigma, y)\, dy\, d\sigma \\ &\geqslant -\frac{2C(1+x)}{\varepsilon} \int_s^t \int_x^{x+\varepsilon} f^\varepsilon(\sigma, y)\, dy\, d\sigma \end{aligned}$$

Dividing the above inequality by $(t-s)$ and letting $s \to t$ yield (5.15). ∎

We next use Lemma 5.3 to derive a lower bound for F^ε. In fact (5.15) reads

$$\partial_t F^\varepsilon(t, x) \geqslant D(1+x)\,\partial_x F^\varepsilon(t, x) + O(\varepsilon)$$

and the above inequality without the $O(\varepsilon)$ term would yield

$$F^\varepsilon(t, x) \geqslant F^\varepsilon(0, (1+x)\,e^{Dt} - 1)$$

by direct integration. Owing to the $O(\varepsilon)$ term a less precise result is available but is still sufficient for our purpose.

Lemma 5.4. Let $T \in \mathbb{R}_+$. There is a positive real number $L(T)$ depending only on a, b, λ, μ, f^{in} and T such that there holds

$$F^\varepsilon(t, x) \geqslant G^\varepsilon(t, x) := F^\varepsilon(0, (1+x)\,e^{Dt} - 1) - \varepsilon L(T)(1+x)(e^{Dt} - 1) \tag{5.16}$$

for $t \in [0, T]$ and $x \in [3\varepsilon, +\infty)$, where D is defined in Lemma 5.3.

Proof. Consider $t \in [0, T]$ and $x \in [3\varepsilon, +\infty)$. The definition of $G^\varepsilon(t, x)$ entails that

$$\begin{aligned}
&\partial_t G^\varepsilon(t, x) + \frac{D(1+x)}{\varepsilon}\,(G^\varepsilon(t, x) - G^\varepsilon(t, x+\varepsilon)) \\
&\quad = \frac{D(1+x)\,e^{Dt}}{\varepsilon} \int_0^\varepsilon [f^\varepsilon(0, (1+x+y)\,e^{Dt} - 1) - f^\varepsilon(0, (1+x)\,e^{Dt} - 1)]\,dy \\
&\qquad - D(1+x)\,L(T)
\end{aligned}$$

By (2.35) we have

$$|f^\varepsilon(0, y) - f^\varepsilon(0, z)| \leqslant 2\,|\partial_x f^{in}|_{L^1}$$

for $y \geqslant 3\varepsilon$ and $z \geqslant 3\varepsilon$. Since $x \geqslant 3\varepsilon$ we also have $(1+x+y)\,e^{Dt} - 1 \geqslant 3\varepsilon$. Therefore

$$\begin{aligned}
&\partial_t G^\varepsilon(t, x) + \frac{D(1+x)}{\varepsilon}\,(G^\varepsilon(t, x) - G^\varepsilon(t, x+\varepsilon)) \\
&\quad \leqslant D(1+x)(2e^{Dt} - L(T))
\end{aligned}$$

We now choose $L(T) = 2e^{DT}$ so that

$$\partial_t G^\varepsilon(t, x) + \frac{D(1+x)}{\varepsilon}\,(G^\varepsilon(t, x) - G^\varepsilon(t, x+\varepsilon)) \leqslant 0 \tag{5.17}$$

for $t \in [0, T]$ and $x \in [3\varepsilon, +\infty)$. Consequently, by (5.15) and (5.17) we have

$$\partial_t (G^\varepsilon - F^\varepsilon)_+ (t, x) + \frac{D(1+x)}{\varepsilon} (G^\varepsilon - F^\varepsilon)_+ (t, x)$$
$$\leqslant \frac{D(1+x)}{\varepsilon} (G^\varepsilon - F^\varepsilon)_+ (t, x+\varepsilon)$$

for $t \in [0, T]$ and $x \in [3\varepsilon, +\infty)$, where $r_+ = \max\{r, 0\}$ denotes the positive part of the real number r. Let $R \geqslant 1$ and integrate the above inequality over $(0, t) \times (3\varepsilon, R)$, $t \in (0, T)$. Since $G^\varepsilon(0, x) = F^\varepsilon(0, x)$ we obtain

$$\int_{3\varepsilon}^{R} (G^\varepsilon - F^\varepsilon)_+ (t, x)\, dx \leqslant \frac{D}{\varepsilon} \int_0^t \int_R^{R+\varepsilon} (1+x)(G^\varepsilon - F^\varepsilon)_+ (s, x)\, dx\, ds \qquad (5.18)$$

But notice that

$$(G^\varepsilon - F^\varepsilon)(t, x) \leqslant |f^{in}|_{L^1} - \varepsilon L(T)(1+x)(e^{Dt} - 1) \leqslant 0$$

for x large enough whence

$$(G^\varepsilon - F^\varepsilon)_+ (t, x) = 0$$

for x large enough. Taking R sufficiently large in (5.18) ensures that the right-hand side of (5.18) is equal to zero from which we deduce (5.16). ∎

We are now in a position to obtain the expected boundedness of (u^ε).

Proposition 5.5. Let $T \in \mathbb{R}_+$. There are two constants $C(T)$ and $\varepsilon(T) \in (0, 1)$ depending only on a, b, λ, μ, f^{in} and T such that

$$u^\varepsilon(t) \leqslant C(T) \quad \text{for} \quad t \in [0, T] \quad \text{and} \quad \varepsilon \in (0, \varepsilon(T)) \qquad (5.19)$$

Proof. By (5.14) and (5.16) we have

$$A^\varepsilon(t) \geqslant a(e^{-Dt} - \varepsilon)^\lambda F^\varepsilon(t, e^{-Dt})$$
$$\geqslant a(e^{-DT} - \varepsilon)^\lambda (F^\varepsilon(0, e^{DT}) - \varepsilon L(T)\, e^{DT})$$

If $\varepsilon \leqslant e^{-DT}/2$ we further obtain

$$A^\varepsilon(t) \geqslant C(T)(F^\varepsilon(0, e^{DT}) - \varepsilon L(T)\, e^{DT})$$

while (2.35) and (2.37) ensure that

$$F^\varepsilon(0, e^{DT}) \geqslant \int_{e^{2DT}}^{\infty} f^{in}(x)\, dx > 0$$

Therefore there are $\varepsilon(T) \in (0, 1)$ and $\delta(T) > 0$ such that

$$C(T)\left(\int_{e^{2DT}}^{\infty} f^{in}(x)\,dx - \varepsilon L(T)\,e^{DT}\right) \geqslant \delta(T)$$

for each $\varepsilon \in (0, \varepsilon(T))$. Combining the previous three inequalities yields that $A^{\varepsilon}(t) \geqslant \delta(T) > 0$ for $t \in [0, T]$ and $\varepsilon \in (0, \varepsilon(T))$. Recalling (5.10) and (5.12) we conclude that u^{ε} satisfies the following differential inequality

$$\varepsilon^{\lambda-\mu}\,\frac{du^{\varepsilon}}{dt}(t) + \delta(T)\,u^{\varepsilon}(t) \leqslant C(T) \qquad \text{for} \quad t \in [0, T] \quad \text{and} \quad \varepsilon \in (0, \varepsilon(T))$$

whence

$$u^{\varepsilon}(t) \leqslant u^{\varepsilon}(0)\,\exp\left\{-\frac{\delta(T)\,t}{\varepsilon^{\lambda-\mu}}\right\} + \frac{C(T)}{\delta(T)}$$

for $t \in [0, T]$. The assertion (5.19) then follows at once from the above inequality since $u^{\varepsilon}(0) = \varepsilon^{\mu-\lambda} c_1^{in,\varepsilon}$, $c_1^{in,\varepsilon} \leqslant \varepsilon\,|f^{in}|_{L^1}$ and $\lambda \in (0, 1]$. ∎

Proof of Theorem 2.5. Owing to Proposition 5.5 the proof of Theorem 2.5 follows the lines of that of Theorem 2.2 as already mentioned. In particular, since the functions $k(x) = ax^{\lambda}$ and $q(x) = bx^{\mu}$, $x \in \mathbb{R}_+$, fulfil the assumptions (2.19), (2.20) and (2.21) a statement similar to (4.2) holds true and we may proceed as in the previous section to obtain a sequence (ε_n) of real numbers in $(0, 1)$, $\varepsilon_n \to 0$, and a function

$$f \in \mathscr{C}([0, +\infty); w-L^1(\mathbb{R}_+, x\,dx)) \tag{5.20}$$

such that

$$f^{\varepsilon_n} \longrightarrow f \qquad \text{in} \quad \mathscr{C}([0, T]; w-L^1(\mathbb{R}_+, x\,dx)) \tag{5.21}$$

for each $T \in \mathbb{R}_+$. Clearly (5.21) ensures that $f(t)$ is non-negative almost everywhere in $\mathbb{R}_+$ for $t \geqslant 0$ and it readily follows from (5.9) and (5.21) that $f \in L^{\infty}(0, T; L^1(\mathbb{R}_+))$ for $T \in \mathbb{R}_+$. Owing to Proposition 5.5 we may also assume that there is a function $u \in L^{\infty}(0, T)$ such that

$$u^{\varepsilon_n} \overset{*}{\rightharpoonup} u \qquad \text{in} \quad L^{\infty}(0, T) \tag{5.22}$$

for each $T \in \mathbb{R}_+$. Thanks to (5.7), (5.20) and (5.22) we may pass to the limit as $\varepsilon_n \to 0$ in (5.4) and (5.3) and conclude that f satisfies (2.44) and (2.46) which in turn implies that u is given by (2.45). The proof of Theorem 2.5 is therefore complete. ∎

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